Assignment 10.

This homework is due *Thursday*, November 7.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 6.

1. Quick reminder

(P) For an arbitrary *nonnegative* measurable function $f : E \to \mathbb{R} \cup \pm\infty$, define its Lebesgue integral by

$$\int_E f = \sup \left\{ \int_E h \;\middle|\; h \text{ bounded, measurable, of finite support and } 0 \le h \le f \text{ on } E \right\}$$

(G) Further, for an arbitrary measurable function $f : E \to \mathbb{R} \cup \pm\infty$, define its Lebesgue integral over E by

$$\int_E f = \int_E f^+ - \int_E f^-, \text{ provided at least one of values } \int_E f^+, \int_E f^- \text{ is finite.}$$

In the case when $\int_E f$ is finite (i.e. both $\int_E f^+, \int_E f^-$ are finite) the function f is said to be Lebesgue integrable over E.

Both integrals defined above in (P) and (G) are linear, monotone and domain additive. Key statements about Lebesgue integral are:

Fatou's Lemma. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then $\int_E f \le \liminf \int_E f_n$.

Monotone Convergence Theorem. Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then $\int_E f = \lim \int_E f_n$.

The Lebesgue Dominated Convergence Theorem. Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g integrable over E s.t. $|f_n| \le g$ on E for all n. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim \int_E f_n = \int_E f$.

2. Exercises

(1) (∼4.3.21)
  (a) Let the function f be nonnegative and integrable over E and $\varepsilon > 0$. Show there is a simple function η on E that has finite support, $0 \le \eta \le f$ on E and $\int_E |f - \eta| < \varepsilon$.
  (b) Further, if E is a bounded interval, show that there is a *step* function h on E s.t. $\int_E |f - h| < \varepsilon$. (Reminder: a step function is a function of the form $\sum_{k=1}^n \lambda_k \chi_{I_k}$, where I_k are intervals.)

(2) (4.3.22+) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to f and f be integrable over $\mathbb{R}$. Applying the Fatou's Lemma to integrals over E and $\mathbb{R} \setminus E$, show that

$$\text{if } \int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n, \text{ then } \int_E f = \lim_{n\to\infty} \int_E f_n \text{ for any measurable set } E.$$

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(3) (4.3.23) Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n+1$. Show that $\int_E f = \sum_{n=1}^{\infty} a_n$ using the Monotone convergence theorem.

(4) (4.3.26) Show that the Monotone convergence theorem may not hold for decreasing sequences of functions.

(5) (4.4.29+) For a locally bounded (therefore bounded on bounded sets by Heine–Borel) measurable function f on $[1, \infty)$, define $a_n = \int_n^{n+1} f$ for each $n \in \mathbb{N}$.
   (a) Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges?
   (b) Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely? (*Hint:* Still no.)
   (c) Is the assertion in the previous item true if we additionally require f to be nonnegative on $[1, \infty)$? (*Hint:* Use the Monotone Convergence Theorem.)

(6) (a) (4.4.34) Let f be a *nonnegative* measurable function on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbb{R}} f.$$

   (*Hint:* Use Monotone Convergence theorem.)
   (b) Prove that the same equality holds if f is arbitrary *integrable* over $\mathbb{R}$ function. (*Hint:* Use the Dominated Convergence.)

(7) (4.5.37) Let f be integrable function on E. Show that for each $\varepsilon > 0$, there is a natural number N for which if $n \geq N$, then $\left|\int_{E_n} f\right| < \varepsilon$ where $E_n = \{x \in E \mid |x| \geq n\}$. (*Hint:* Use continuity of integration; or countable domain additivity of integration.)

(8) (4.5.38i) Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = (-1)^n/n$ for $n \leq x < n+1$, $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} \int_1^n f$ exists while f is not integrable over $[1, \infty)$. Does this contradict continuity of integration?